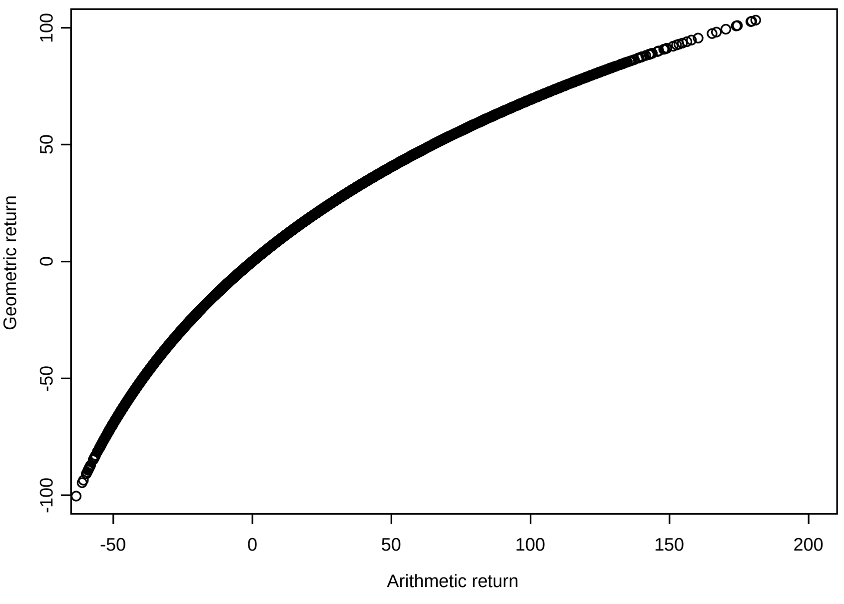
To log or not to log: The distribution of asset returns



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Title: To log or not to log: The distribution of asset returns

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Abstract:
In the context of the measurement of market risk, the random variable is taken as the rate of return of a financial asset. One may define the return in different ways, the two most common are arithmetic and geometric returns. The distinction between these two types of returns is not well understood. They are frequently assumed to be approximately equal. Moreover they both are assumed to be normally distributed. In this paper we explain the difference. We show that both types cannot be normally distributed, and that the difference grows larger as the volatility of the financial asset increases and the time resolution decreases.

Keywords: Arithmetic return, geometric return, normal distribution, lognormal distribution

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1 Introduction

In the context of the measurement of market risk, the random variable is taken as the rate of return of a financial asset. One may define several kinds of returns, of which the two most common are arithmetic and geometric returns. In Section 2 we explain the difference between the two types. It is common to assume that single-period geometric returns follow a normal distribution. In Section 3 we discuss what this means for the distribution of multi-period geometric returns, and what it means for the distribution of arithmetic returns. Asset classes are often combined into portfolios and there is a need to compute the portfolio return. Section 4 treats the computation of the arithmetic and geometric portfolio return, and relates them to the mean-variance approach of Markowitz. Finally, in Section 5, we give some simulation examples, showing that the differences between arithmetic and geometric return distributions will grow larger as the volatility of the returns increases. We also study the distributional properties of historical annual arithmetic and geometric returns from different markets.

2 Definitions

There are two main kinds of returns, *arithmetic returns* and *geometric returns* (Jorion, 1997). Daily arithmetic returns are defined by

$$r_t = (V_t - V_{t-1})/V_{t-1},$$

where V_t is the price of the asset at day t. Yearly arithmetic returns are defined by

$$R = (V_T - V_0)/V_0,$$

where V_0 og V_T are the prices of the asset at the first and last trading day of the year, respectively. We have that R may be written as

$$R = \frac{V_T}{V_0} - 1 = \frac{V_T}{V_{T-1}} \frac{V_{T-1}}{V_{T-2}} \cdots \frac{V_1}{V_0} - 1 = \prod_{t=1}^{T} \frac{V_t}{V_{t-1}} - 1,$$

that is, it is not possible to describe the yearly arithmetic return as a function or sum of daily arithmetic returns.

Daily geometric returns are defined by

$$d_t = \log(V_t) - \log(V_{t-1}),$$

while yearly geometric returns are given by

$$D = \log(V_T) - \log(V_0).$$

We have that D may be written as

$$D = \log\left(\prod_{t=1}^{T} \frac{V_t}{V_{t-1}}\right) = \sum_{t=1}^{T} \log\left(\frac{V_t}{V_{t-1}}\right) = \sum_{t=1}^{T} d_t,$$

which means that yearly geometric returns are equal to the sum of daily geometric returns.

The relationship between yearly geometric and arithmetic returns is given by

$$D = \log(1 + R).$$

Hence, D can be decomposed into a Taylor series as

$$D = R + \frac{1}{2}\,R^2 + \frac{1}{3}\,R^3 + \cdots\,,$$

which simplifies to R if R is small. Thus, when arithmetic returns are small (i.e. on daily resolution), there will be little difference between geometric and arithmetic returns.

3 Distribution

It is very common to assume that geometric returns have constant expectation and variance, that they are serially independent, and that they are normally distributed on all time resolutions. The purpose of this note is not to defend these assumptions. It is widely known that the volatility of geometric returns from financial market variables such as exchange rates, equity prices, and interest rates measured over short time intervals (i.e. daily or weekly) vary over time. Moreover, these returns are known to have a distribution that is more peaked and has fatter tails than the normal distribution. Nevertheless, the assumptions of normal distribution and constant mean and variance are standard in financial analysis. Black & Scholes' formula for options (Black and Scholes, 1973) is for instance based on them. Hence, it is important to know what they really mean in different situations.

For longer time periods, the Central Limit Theorem (Lindeberg, 1922) is often invoked in defense of the normal distribution. Even if the daily returns are non-normal, the Central Limit Theorem tells us that the sum of N independent, identically distributed random variables with finite variance converges to a normal distribution when N is large. From a probabilistic point of view, it is not at all obvious that the assumptions of the Central Limit Theorem are satisfied. In reality, geometric returns are uncorrelated, but not independent. The greater the dependence, the worse the normal approximation. This affects the speed of the convergence. It is usually most important that the approximation holds in the tails. However, even when the conditions for the Central Limit Theorem hold, the convergence in the tails is known to be very slow (Bradley and Taqqu, 2003). The normal approximation may then only be valid in the central part of the distribution. Finally, the Central Limit Theorem assumes identically distributed variables, while the the volatility of geometric returns is known to vary over time.

If one defines arithmetic yearly returns to be normally distributed, daily arithmetic returns will not be normally distributed, and if one defines daily arithmetic returns to be normally distributed, yearly arithmetic returns will not be normally distributed. This can be seen from the definition of R in Section 2 and the simple fact that a product of normally distributed variables is not normally distributed.

If a variable $X = \log(Y)$ is Gaussian distributed, it follows that Y is lognormally distributed. Hence, from the relationship between arithmetic and geometric returns specified in Section 2, we have that arithmetic returns will follow a lognormal distribution if the geometric returns follow a normal distribution. The lognormal distribution will be skewed to the right, due to the relationship between the lognormal and normal distributions, and the symmetry of the latter. In what follows we describe the relationship between the expectation and variance of the two distributions.

Assume that

$$X = \log Y \sim N(\mu, \sigma^2).$$

We then have that

$$\mathrm{E}(Y) = \exp(\mu + \sigma^2/2) \tag{1}$$

and

$$\mathrm{Var}(Y) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1). \tag{2}$$

In addition to the fact that compounded geometric returns are given as sums of geometric returns, there is another advantage of working with the log-scale. If the geometric returns are normally distributed, we will never get negative prices. In contrast, assuming that arithmetic returns are normally distributed may lead to negative prices, which is economically meaningless.

4 Portfolio return

Asset classes are often combined into portfolios and there is a need to compute the portfolio return over a period of length T. Let $V_{0,P}$ and $V_{T,P}$ be the value of the portfolio at the start and end of the period. Then we have that

$$V_{T,P} = \sum_i V_{T,i} = \sum_i (V_{T,i} - V_{0,i} + V_{0,i}) = \sum_i V_{0,i}\left(1 + R_i\right),$$

where $V_{t,i}$ is the value of asset i at time t, and R_i is the arithmetic return of asset i over the period. The arithmetic return of the portfolio over the period is

$$R_P = \frac{V_{T,P} - V_{0,P}}{V_{0,P}} = \frac{\sum_i V_{0,i}\left(1 + R_i\right) - \sum_i V_{0,i}}{\sum_i V_{0,i}} = \frac{\sum_i V_{0,i}\, R_i}{\sum_i V_{0,i}} = \sum_i \frac{V_{0,i}}{V_{0,P}} R_i,$$

i.e. the arithmetic return of a portfolio is a weighted average of the component asset arithmetic returns.

The same is not true for the geometric returns. Let D_P be the geometric return of the portfolio over the period of length T. We have that

$$\exp(D_P) = \left(\frac{V_{T,P}}{V_{0,P}}\right) = \left(\frac{\sum_i V_{T,i}}{V_{0,P}}\right) = \left(\frac{\sum_i V_{0,i}\exp(D_i)}{V_{0,P}}\right) = \sum_i \frac{V_{0,i}}{V_{0,P}} \exp(D_i),$$

where D_i is the geometric return of asset i over the period. This means that $D_P = \log\left(\sum_i \frac{V_{0,i}}{V_{0,P}} \exp(D_i)\right)$ which can be shown to be greater than the weighted average of the component asset geometric returns.

Markowitz portfolio theory (Markowitz, 1952) uses the fact that the return of the portfolio can be written as a weighted average of the component asset returns. From what we have shown above, this means that this theory is based on arithmetic returns. In the Markowitz framework (commonly denoted the mean-variance approach), the aim is, for a given upper bound of the portfolio variance, to select a portfolio that maximizes the portfolio return, or equivalently, for a given lower bound of the portfolio return, to select a portfolio that minimizes the portfolio variance. In practice, the mean-variance approach assumes that the risk of the portfolio can be totally described by its variance. If the arithmetic returns were multivariate normally distributed, this would be true. However, as shown in Section 3, the arithmetic return of each asset is lognormally distributed, and moreover they are dependent, meaning that we do not even have an analytical expression for the distribution of their weighted sum. Approximating this distribution with another lognormal distribution (which might be a crude approximation !) implies that the portfolio arithmetic return distribution is skewed, with lighter left tail and heavier right tail than the normal distribution. Hence, instead of the standard deviation, it would be better to use other risk measures, as for instance the VaR (Jorion, 1997) and expected shortfall (Artzner et al., 1997), when determining the optimal portfolio.

5 Examples

If the volatility of the price series is small, and the time resolution is high, the difference between arithmetic and geometric returns will be small. However, as the volatility increases and the time resolution decreases, the difference grows larger. To illustrate this effect, we have simulated daily geometric returns from two different normal distributions. The first corresponds to expected annual arithmetic return and standard deviation of 8.1% and 26.4%, respectively, while the corresponding quantities for the other are 6.4% and 5.0%. Based on the simulations, we have generated the distributions for annual geometric and arithmetic returns. Figure 1 shows the densities corresponding to the largest standard deviation. The lognormal distribution of the arithmetic returns is clearly skewed to the right. Figure 2 shows the densities corresponding to the smallest standard deviation. Here, the difference between the normal and the lognormal distribution is smaller. Figures 3 and 4 shows QQ-plots of the arithmetic returns versus the geometric returns for the two cases. The QQ-plot corresponding to the largest volatility has a significant exponential shape, while the other shows a curve that is closer to a straight line.

We have also studied historical data from the Norwegian, American, German and Japanese stock markets. For all markets we have 33 independent observations corresponding to the years 1970 to 2002. Figures 5–8 show the QQ-plots of the annual geometric returns against the normal distribution for the four markets. For the Norwegian and Japanese market, the fit to the normal distribution is quite good in the left tail of the distribution, but not so good in the right tail. For the German and the American market, the fit is relatively good in the right tail, but not so good in the left. The plots show

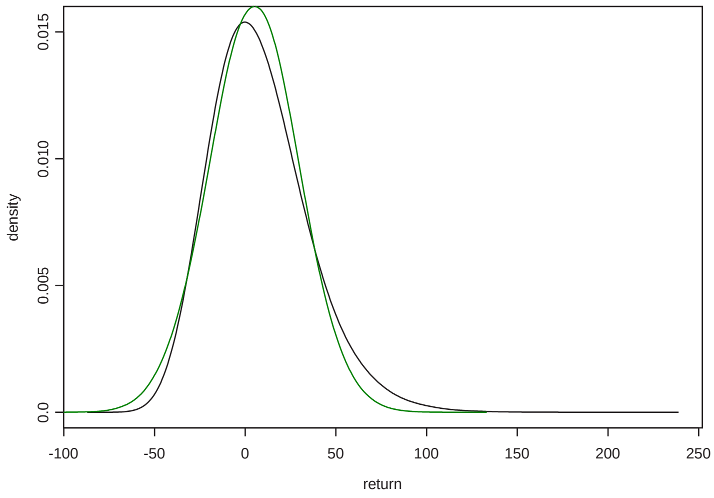

Figure 1: *Density for arithmetic return (black line) and geometric return (green line) corresponding to annual arithmetic return and standard deviation of 8.1% and 26.4%.*

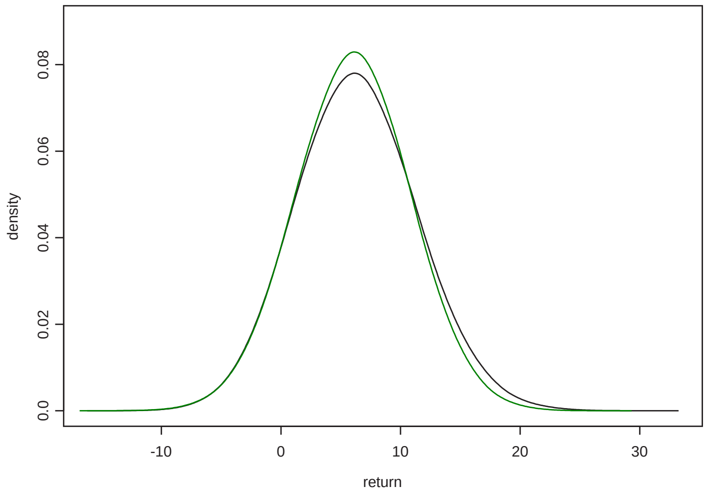

Figure 2: *Density for arithmetic return (black line) and geometric return (green line) corresponding to annual arithmetic return and standard deviation of 6.4% and 5.0%.*

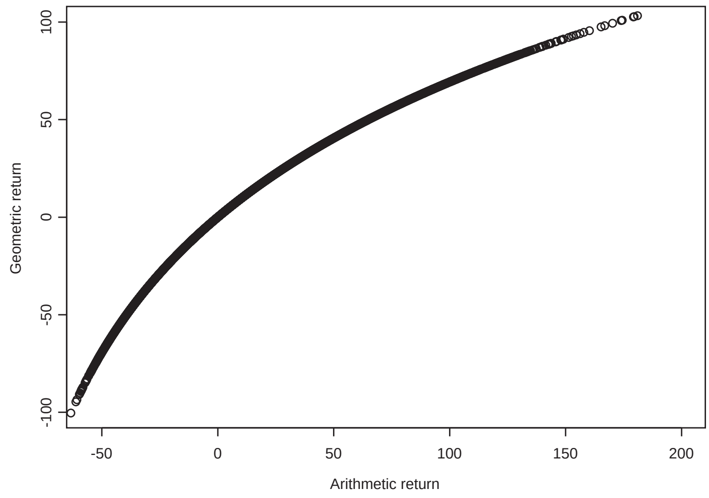

Figure 3: *Arithmetic returns vs. geometric returns corresponding to annual arithmetic return and standard deviation of 8.1% and 26.4%.*

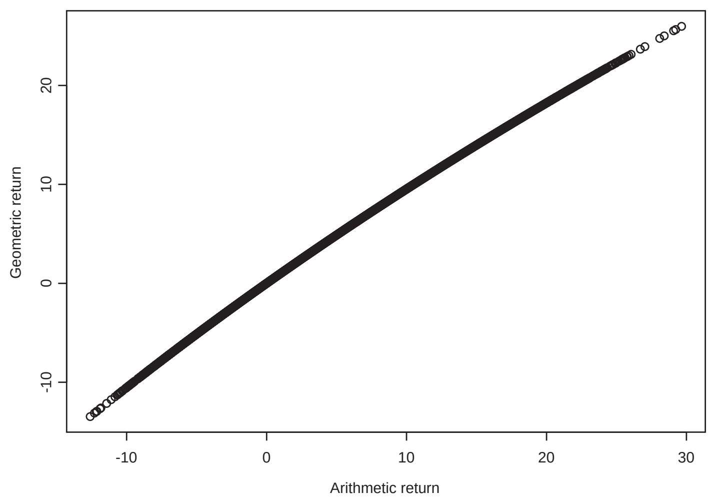

Figure 4: *Arithmetic returns vs. geometric returns corresponding to annual arithmetic return and standard deviation of 6.4% and 5.0%.*

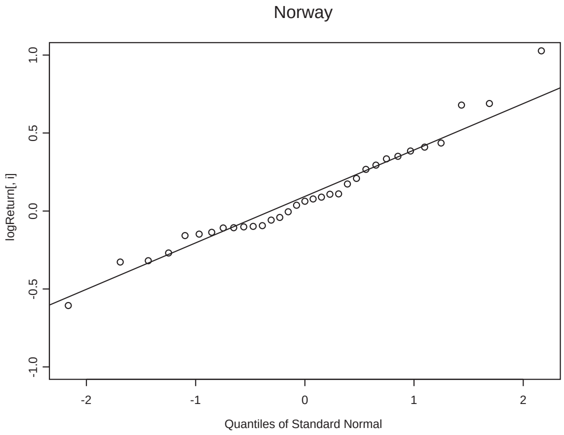

Figure 5: *Norway: Annual geometric historical returns against the normal distribution.*

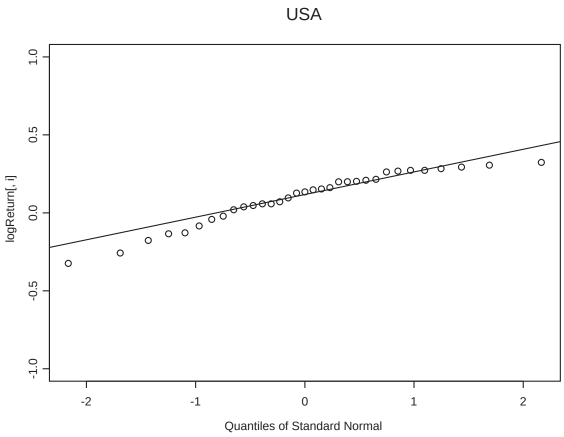

Figure 6: *USA: Annual geometric historical returns against the normal distribution.*

that summing 250 daily log returns is not enough for the Central Limit Theorem to work when the daily returns are heavy-tailed, serially dependent and not identically distributed. Figures 9-12 show QQ-plots of the annual geometric returns against the annual arithmetic returns for the four markets. For the Norwegian market there seems to be a quite large difference between the arithmetic and the geometric distributions for the annual returns. For the American market, the difference seems to be quite small, while it is somewhat in between for the Japanese and German markets. This can be explained by the historical annual arithmetic volatility, which has been 44%, 18%, 27% and 29%, respectively, for the Norwegian, American, German and Japanese stock markets during the time period we have studied.

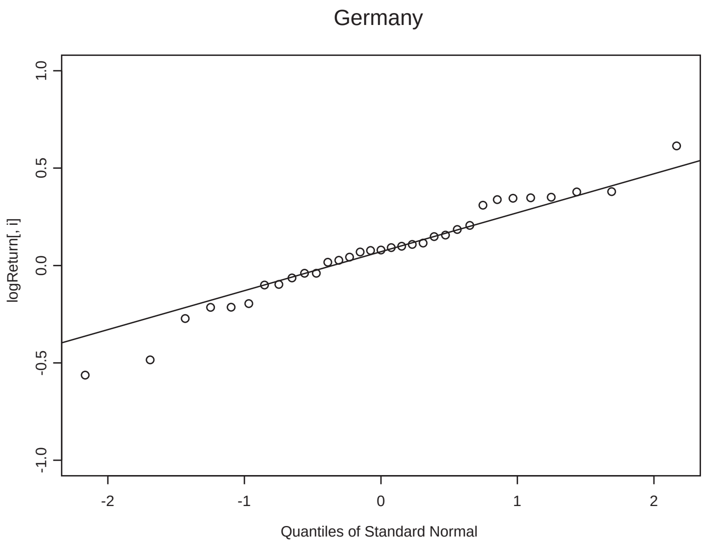

Figure 7: *Germany: Annual geometric historical returns against the normal distribution.*

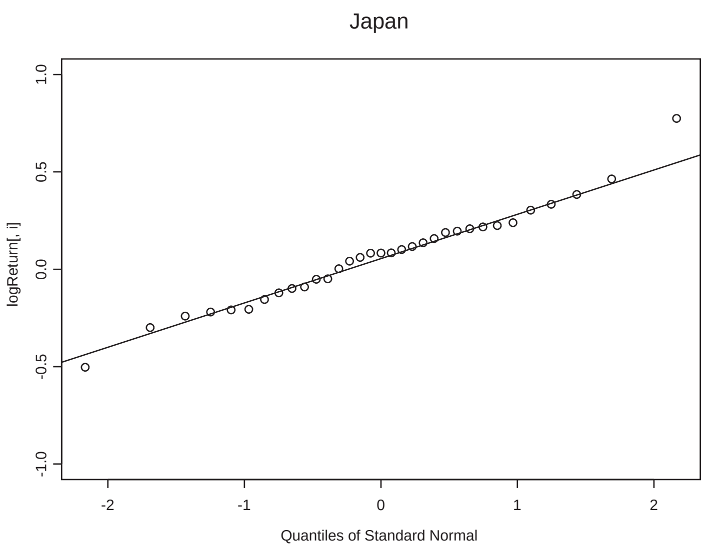

Figure 8: *Japan: Annual geometric historical returns against the normal distribution.*

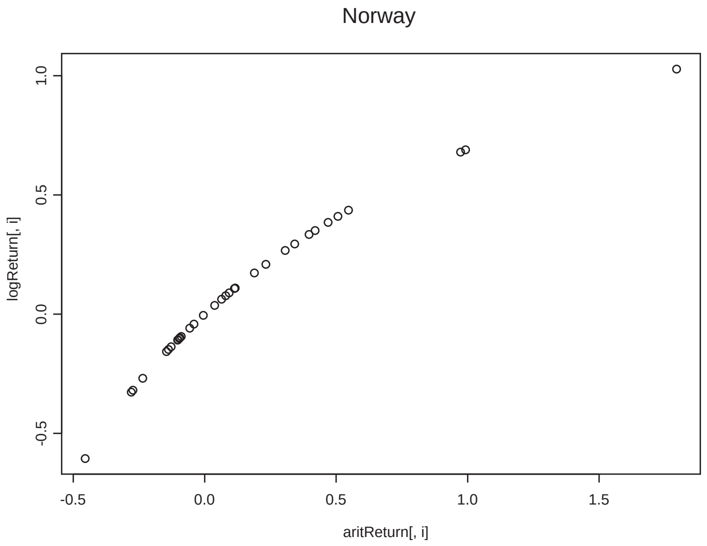

Figure 9: *Norway: Annual geometric historical returns against annual arithmetic historical returns.*

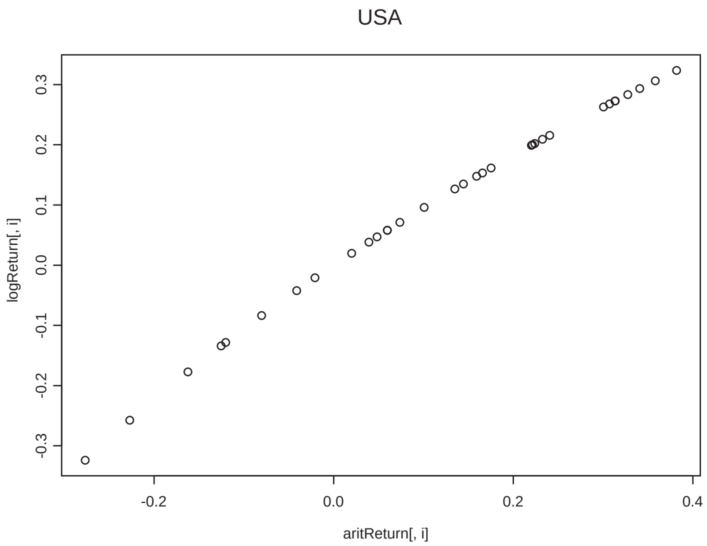

Figure 10: *USA: Annual geometric historical returns against annual arithmetic historical returns.*

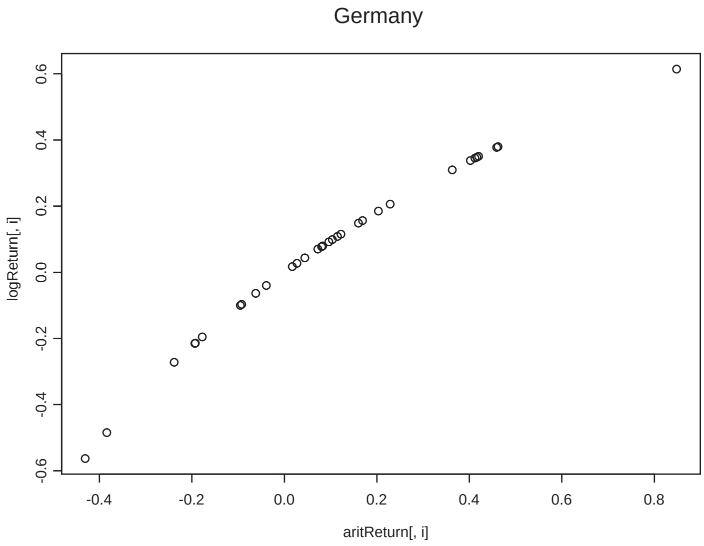

Figure 11: *Germany: Annual geometric historical returns against annual arithmetic historical returns.*

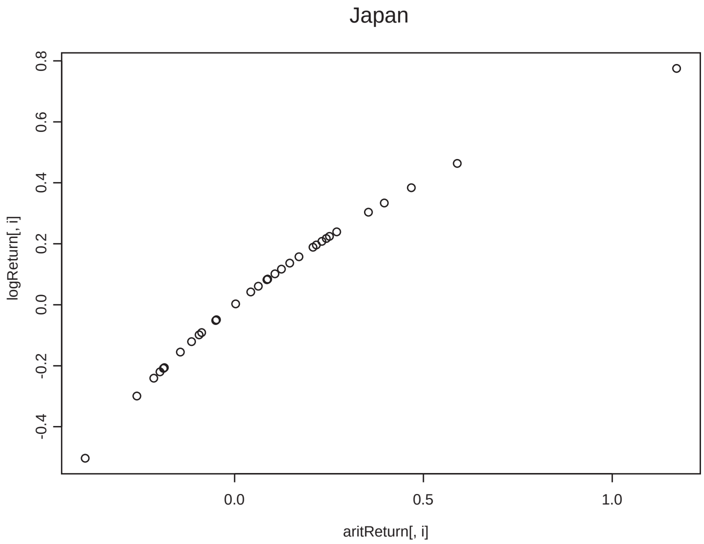

Figure 12: *Japan: Annual geometric historical returns against annual arithmetic historical returns.*

6 Summary

The distinction between arithmetic and geometric returns is not well understood. Both are for instance frequently assumed to be normally distributed. The Black and Scholes formula for option pricing assumes that geometric returns are normally distributed on all time resolutions, while the famous Markowitz portfolio theory is built on the assumption of normally distributed arithmetic returns. We have in this note shown that both kinds of returns cannot be normally distributed, and that the difference grows larger as the volatility of the financial asset increases and the time resolution decreases.

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