SOME PROPERTIES OF EULER CAPITAL ALLOCATION

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ABSTRACT. The paper discusses capital allocation using the Euler formula and focuses on the risk measures Value-at-Risk (VaR) and Expected shortfall (ES). Some new results connected to this capital allocation is shown. Two examples illustrate that capital allocation with VaR is not monotonous which may be surprising since VaR is monotonous. A third example illustrates why the same risk measure should be used in capital allocation as in the evaluation of the total portfolio. We show how simulation may be used in order to estimate the expected Return on risk adjusted capital in the commitment period of an asset. Finally, we show how Markov chain Monte Carlo may be used in the estimation of the capital allocation.

1. INTRODUCTION

The regulatory framework requires a quantification of the total risk in a corporate. Based on the quantification and a risk measure there is a requirement for an economic capital that is able to absorb potential losses. The competition makes it necessary for a financial corporate to ensure an efficient use of their economic capital. As a part of improving the use of the economic capital, the capital that is allocated to each asset is calculated. This makes it possible to evaluate each part of the portfolio. This paper studies some properties of capital allocation assuming that the stochastic properties of the different assets including their correlations are known. Hence, we will not discuss how to estimate these distributions.

The two papers Artzner et al. [2] and [3] have initiated a large number of papers giving a much better understanding of capital allocation. There seem to be an agreement that capital should be allocated proportional with the partial derivatives of the risk measure and use the Euler Theorem, see Tasche [12]. This was first proposed by Litterman [7]. Denault [4] argues for the same capital allocation using theory from cooperative games. Kalkbrener et al., [6] shows how axioms assuming that the capital allocation is linear and diversifying lead to the same capital allocation. Tasche [13] and [14] and Fischer [5] give an overview of the argument for using Euler allocation.

A large number of authors, see e.g. Acerbie et al [1], argue that the risk measure should be coherent. This includes for example expected shortfall (ES) but excludes Value-at-Risk (VaR), the most used risk measure. VaR is used as a part of the Basel II framework. VaR is not subadditive implying that there may be an additional cost when adding two portfolios instead of a saving due to diversification. Several authors argue that one should use ES which is the smallest measure that is law-permitting dominating VaR, see Tasche [12].

We will contribute in this area with several different examples. First illustrating that capital allocation using VaR is not monotonous. This may be surprising since VaR is monotonous. It may be a larger problem than the missing subadditivity of VaR in a regular use of VaR in capital allocation in corporate. This easily leads to suboptimal performance of the management of the portfolio.

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The regulatory framework may require that VaR is used for estimating the economic capital needed for a portfolio. It may however be tempting to use ES for the capital allocation of each asset since this is a coherent measure with better mathematical properties. We give an example illustrating that the same risk measure should be used for the total portfolio and the capital allocation, else one easily gets conflict of interest.

Each part of the portfolio may be a commitment for a longer period, and the corporate will not always be able to end the part of the portfolio that is not cost efficient based on the latest evaluations. Hence, a new investment should be evaluated based on the expected capital requirement in the commitment period for the investment, not only based on today's portfolio. Capital allocation based on the present portfolio is not optimal since the total portfolio may change during the period where it is not possible to change the condition for a part of the portfolio. The natural alternative is to simulate the portfolio and use the expected average value of the economic capital for the commitment period of the new investment. This is illustrated in an example.

The numerical calculations involved in a capital allocation may be very computer intensive. Direct Monte Carlo simulation may not work since we need a large number realizations to make a good evaluation of the tail behavior. Importance sampling reduces the problem, see Kalkbrener et al. [6], but it may still be too CPU requiring. We propose to use Markov chain Monte Carlo. Then we may focus such that all realizations are used in the quantification. This is illustrated in an example.

2. Model

In the following we use the terminology from Tasche [14]. Let the random variable X_i denote the cash flow from asset i. A portfolio X consists of n different assets

$$X = \sum_{i=1}^{n} X_i.$$

Further, let $EC(X) = \rho(X)$ denote the economic capital that is deemed necessary by the regulator or corporate in order to handle a possible unfortunate development of the portfolio. $\rho(X)$ is a risk measure on the portfolio $X \in V$ where V is the set of real valued random variables. We may define ρ as VaR, ES, the standard deviation multiplied with a constant or any other measure for the uncertainty. We need to define some properties on risk measures. A risk measure is *monotonous* if $X, Y \in V$ and $X \leq Y$ almost everywhere implies

$$\rho(X) \geq \rho(Y),$$

subadditive if $X, Y \in V$ implies

$$\rho(X+Y) \leq \rho(X) + \rho(Y),$$

positive homogeneous if $X \in V$, $h \in R$, and $h > 0$ implies

$$\rho(hX) = h\rho(X)$$

and *translation invariant* if $X \in V$ and $h \in R$ implies

$$\rho(X+h) = \rho(X) - h.$$

A risk measure is denoted *coherent* if it is monotonous, subadditive, positive homogeneous and translation invariant. In this paper we will assume the risk measure is positive homogeneous but do not require the other properties above.

In order to define the derivative of ρ we introduce the function

$$f(\mathbf{u}) = \rho(\sum_{i=1}^{n} u_i X_i)$$

where $\mathbf{u} = (u_1, \cdots, u_n)$. Assuming the risk measure is positive homogeneous, it satisfies the Euler formula, see Tasche [12]

$$f(\mathbf{u}) = \sum_{i=1}^{d} u_i \frac{\partial f}{\partial u_i}(\mathbf{u})$$

where we assume the partial derivatives exists. This makes it natural to allocate the capital

$$\rho(X_i, X) = \frac{\partial f}{\partial u_i}(\mathbf{u}_1)$$

where $\mathbf{u}_1 = (1, \cdots, 1)$ to the asset X_i. This capital allocation is denoted Euler allocation. We see immediately that this capital allocation is linear, e.g.

$$\rho(X_i + X_j, X) = \rho(X_i, X) + \rho(X_j, X)$$

and has the full allocation property, e.g.

$$\rho(X) = \sum_{i=1}^{n} \rho(X_i, X).$$

Assuming $\rho(X)$ is coherent, Denault [4] has proved using game theory, that Euler allocation is the only allocation satisfying that we always have $\rho(X_i, X) \leq \rho(X_i)$. Kalkbrener et al., [6], has proved a similar result using axioms. This is a critical property closely connected to subadditivity. It states that an asset is allocated less capital as part of a portfolio than alone. Tasche [11] and [14] give an overview of the argument for this capital allocation.

Another measure on the portfolio is the Return on risk adjusted capital,

$$\text{RORAC}(X) = \frac{EX}{EC(X)}.$$

For each asset X_i in a portfolio X we define $\text{RORAC}(X_i, X) = EX_i/\rho(X_i, X)$. A capital allocation is denoted RORAC compatible, see Tasche [11], if there exists $\varepsilon_i > 0$ such that $\text{RORAC}(X_i, X) > \text{RORAC}(X)$ implies

$$\text{RORAC}(X + hX_i) > \text{RORAC}(X)$$

for all $0 < h < \varepsilon_i$. The intuition is that if asset X_i has higher RORAC than the entire portfolio, then increasing this asset in the portfolio increases the RORAC of the portfolio. This is illustrated in Example 3.

Tasche [9] proves that Euler allocation is the only allocation that is RORAC compatible. In the following two sections we will discuss Euler allocation for two of the most popular risk measures.

3. Value-at-Risk, VaR

Defined the α-quantile q_α as

$$q_\alpha(X) = \inf\{z \in R | P(X \leq z) \geq \alpha\}.$$

Then the risk measure VaR is defined as $VaR_\alpha(X) = q_\alpha(-X)$. We typically have $\alpha \geq 0.99$ implying that estimation of VaR only focus on one point in the tail. This makes estimates for VaR much less stable than for example standard deviation.

VaR is monotone, positive homogeneous, and translation invariant. But it is also well-known that VaR is not subadditive since we may have

$$VaR_\alpha(X + Y) > VaR_\alpha(X) + VaR_\alpha(Y).$$

This implies that VaR is not a coherent risk measure. VaR expresses the economical capital necessary to ensure that the probability for a default is less than α. Hence, VaR focuses only on one point in the cumulative distribution of X, the maximum value z where $P(X \geq z) \geq \alpha$. This property is not additive. This implies that if we have two portfolios X and Y the capital requirement may be larger when we add them together than if we keep them separate. This may be an argument for splitting the portfolio and hence the corporate in two. This may seem counter-intuitive as a risk measure and a not wanted property for the corporate for strategic reasons. There are several papers stating that risk measures that are not coherent should not be used. Artzner et al. [3] give three examples with discrete random variables, one illustrating that the measure is not subadditive, another illustrating that VaR fails to recognize concentration of risk and fails to encourage a reasonable allocation of risk between agents. Tasche [10] gives an example with two independent Pareto distributions that does not satisfy subadditivity. Acerbi et al, [1], writes " ..if a measure is not coherent we just choose not to call it a risk measure at all", particular due to the missing subadditive property and since there exists coherent risk measures with satisfactory properties.

The Euler allocation for VaR is

$$VaR_\alpha(X_i, X) = -E\{X_i | X = -VaR_\alpha(X)\} \tag{1}$$

under some smoothness assumptions, see Tasche [9]. Kalkbrener et al. [6] report that capital allocation with VaR may require larger economic capital to an asset then the lowest possible outcome of the variable. But then the capital allocation is using covariance instead of Euler allocation. This will not happen with Euler capital allocation as is seen from (1). It is much easier to perform capital allocation using correlation than Euler allocation. In Section 7 we show how calculation of capital allocation using Euler allocation may be performed efficiently with Markov chain Monte Carlo methods.

VaR satisfies the following monotonicity property: If $P(X \leq z) \geq P(Y \leq z)$ for all values of $z \in R$ then $VaR_\alpha(X) \geq VaR_\alpha(Y)$. However, Euler capital allocation with VaR does not satisfy the same monotonicity e.g. we may have $VaR_\alpha(X, X + Y) \leq VaR_\alpha(Y, X + Y)$. This may be surprising. Missing this property may be more important than missing subadditivity when considering whether the capital allocation inside a corporate is fair or not. To the authors knowledge, this property is not proved earlier. This property is due to the fact that VaR focuses on only one point in the distribution.

We give two examples that Euler capital allocation with VaR is not monotonous. The first example has two independent discrete variables and the second example has stochastic variables that are continuous and dependent.

Example 1 Let X_1 and X_2 be two independent stochastic variables where

$$P(X_1 = 0) = P(X_2 = 0) = 0.9925$$

and

$$P(X_1 = -200) = P(X_2 = -100) = 0.0075.$$

Then

$$\begin{aligned} VaR_{0.99}(X_1) = VaR_{0.99}(X_2) &= 0 \\ VaR_{0.99}(X_1 + X_2) &= 100 \\ VaR_{0.99}(X_1, X_1 + X_2) &= 0 \\ VaR_{0.99}(X_2, X_1 + X_2) &= 100 \end{aligned}$$

Hence, the asset X_2 gets allocated all the risk even though $X_1 \leq X_2$. This may seem surprising and the example is analysed more thoroughly by introducing weights (u_1, u_2) for the two assets, e.g.

$$X(u_1, u_2) = u_1 X_1 + u_2 X_2.$$

Then we have

$$\begin{aligned} VaR_{0.99}(X(u_1, 1)) &= 100u_1 \quad \text{for } 0 \geq u_1 \geq 2 \\ VaR_{0.99}(X(1, u_2)) &= 100 \quad \text{for } u_2 \geq 0.5. \end{aligned}$$

We see that the risk for the portfolio is sensitive to weight u_1 but not u_2 in the interesting region close to $(u_1, u_2) = (1, 1)$, hence it makes sense to reduce the weight of X_1 instead of X_2 if the regulatory requirement is based on $VaR_{0.99}(X)$. This also illustrates that it is possible to cheat in the system. The person responsible for asset X_1 may reduce the capital allocated to this asset if he promises to give a value of 100 to charity if $X_1 = -100$. This makes the distribution of X_1 and X_2 equal implying that they get the same capital allocation.

Example 2 Let $X = 3X_1 + X_2$ where X_1 is symmetric around 0 and X_2 is given as

$$X_2 = \begin{cases} -X_1 & \text{if } X_1 \leq 0 \\ -2X_1 & \text{if } X_1 > 0. \end{cases}$$

This implies that X_2 has the same upside as X_1, but the downside is not as good considered as a univariate variable. But X_2 has good diversification properties in the portfolio. The portfolio X has the same marginal distribution as X_2,

$$X = \begin{cases} 2X_1 & \text{if } X_1 \leq 0 \\ X_1 & \text{if } X_1 > 0. \end{cases}$$

Then $VaR_\alpha(X_2, X) < 0 < VaR_\alpha(X_1, X)$ even though $P(X_1 \leq z) \leq P(X_2 \leq z)$ all $z \in R$.

The situation is very different in the two cases. In Example 1 it is tempting to deviate from Euler capital allocation by moving capital between the involved assets in order to maintain monotonicity. In Example 2 it is important to not change the Euler capital allocation in order to honor the increased diversification.

4. Expected shortfall, ES

ES is defined as

$$ES_\alpha(X) = -E\{X | X \leq -VaR_\alpha(X)\}.$$

Acerbi et al, [1] prove that

$$ES_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_\tau(X) d\tau.$$

ES is a coherent risk measure. Further, it is proved that Euler allocation gives

$$ES_\alpha(X_i, X) = -E\{X_i | X \leq -VaR_\alpha(X)\}$$

assuming sufficient smoothness. Since ES takes the average of VaR for $\alpha \leq \tau \leq 1$ we have $ES_\alpha(X) \leq VaR_\alpha(X)$. It is also proved the ES is the smallest risk measure that is law invariant where $\rho(X) \leq VaR_\alpha(X)$. This makes ES a good alternative to VaR. The property *law invariant* is that if $X, Y \in V$ and $P(X \leq z) = P(Y \leq z)$ for all $z \in R$ then $\rho(X) = \rho(Y)$. Since ES focuses on the entire tail and not only a quantile, it may require more realizations in a Monte Carlo estimation than VaR.

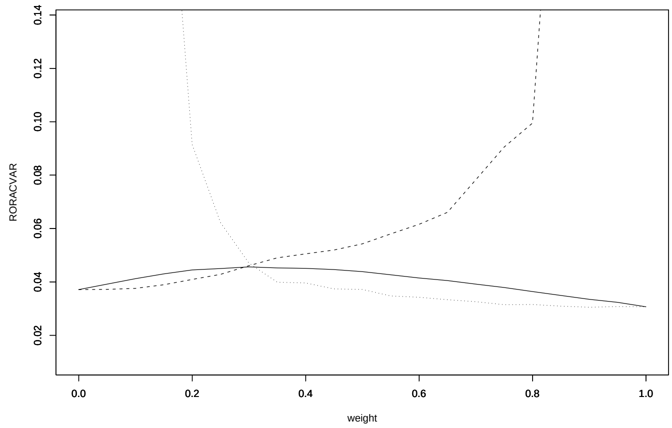

FIGURE 1. RORAC for the porfolio (line) and the assets uX_1 (dotted) and $(1-u)X_1$ (dashed) with Euler capital allocation using VaR.

5. COMBINING ES AND VAR

Since ES is coherent while VaR is not, many authors recommend to use ES instead of VaR. We will discuss under the assumption that the regulator requires the use of VaR for setting the economic capital whether to use VaR or ES in the capital allocation. Since VaR is not subadditive some authors seem to recommend to use ES for capital allocation. The missing monotonicity property strengthen this argument. However, we will argue that this is not necessarily a good choice. It is correct that VaR may lead to examples where we do not get subadditivity and therefore there may be arguments for splitting the portfolio in several parts. However, most practitioners report a 30% diversification effect indicating that the missing subadditivity property is mainly academic. The following example shows that combining VaR as a risk measure for the portfolio and ES in the capital allocation, may give conflict of interest. This is avoided if VaR or ES is used both as a risk measure for the portfolio and in the capital allocation. This indicates that if VaR is used as risk measure, then VaR should also be used in the capital allocation.

Example 3 Let $X_i = 0.5 - I_i Y_i$ for $i = 1, 2$ where I_i is an indicator and Y_i is a Pareto distributed variable with density

$$f(y) = \frac{\gamma_i}{b_i}(\frac{y}{b_i} + 1)^{-\gamma_i - 1}$$

where $y > 0$. The indicator $I_i = 1$ with probability 0.1 and else $I_i = 0$. We have chosen the variables $\gamma_1 = 5$ and $\gamma_2 = 1.7$ and b_i such that $E\{X_i\} = 0.2$ for $i = 1, 2$. We study the portfolio $X = uX_1 + (1-u)X_2$ for $0 \le u \le 1$ and compare RORAC with Euler capital allocation using VaR and ES as risk measure (Figures 1 and 2), and when VaR is used as risk measure for the portfolio while

$$\rho_{VaR-ES}(X_i, X) = ES_\alpha(X_i, X) VaR_\alpha(X)/ES_\alpha(X)$$

is used in the capital allocation (Figure 3). The figures show RORAC for the portfolio and the two assets uX_1 and $(1-u)X_1$ as a function of u for the different risk measures and capital allocation methods. When we use Euler capital allocation with VaR or ES we get an optimal portfolio for $u = 0.3$ and $u = 0.7$ respectively where both assets and the total portfolio have the same RORAC. That the assets have the same RORAC in the optimum as the total portfolio follows from the RORAC

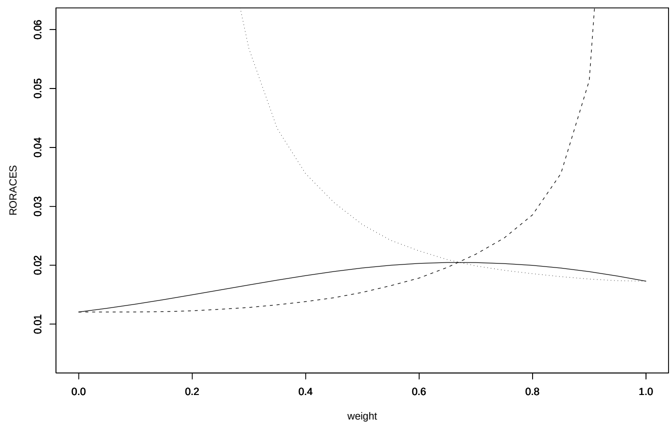

FIGURE 2. RORAC for the porfolio (line) and the assets uX_1 (dotted) and $(1-u)X_1$ (dashed) with Euler capital allocation using ES.

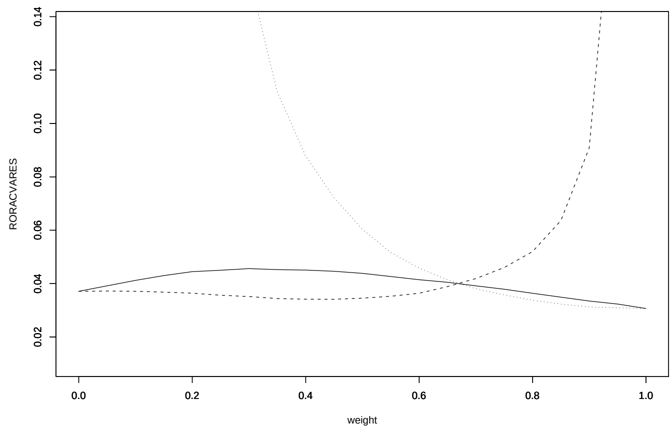

FIGURE 3. RORAC for the porfolio (line) and the assets uX_1 (dotted) and $(1-u)X_1$ (dashed) with VaR as risk measure and $\rho_{VaR-ES}(X_i, X)$ used in the capital allocation.

compatibility of Euler capital allocation. In these two cases RORAC is increasing for the assets when the asset gets less weight in the portfolio. When ρ_{VaR-ES} is used for the capital allocation the performance is more complex. The diversification effect is not as expected and the two assets have very different RORAC at the optimal diversification. If we require that the two assets have approximately the same RORAC then the total portfolio is far from optimal. If we reduce the weight of the asset with lowest RORAC, we may reduces the RORAC of the total portfolio.

TABLE 1. RORAC for a new investment depending on the other investment.

New investment	Other investment	
	X_1	X_2
X_1	0.030	0.033
X_2	0.071	0.044

6. Changes in the portfolio

Assume we already have a portfolio X and consider to expand the portfolio with X_{n+1}. Then we should base the evaluation on the capital allocation

$$\rho(X_{n+1}, X + X_{n+1}). \tag{2}$$

Euler capital allocation does not encourage new investments X_{n+1} where

$$\rho(X, X + X_{n+1}) - \rho(X)$$

is large, hence contributes to a reduction in the capital allocated to the rest of the portfolio. There are capital allocation methods that takes these other properties into account. But Tasche [12] proves that for continuous differentiable, subadditive and positive homogeneous risk measures then Euler allocation gives

$$\rho(X_{n+1}, X + X_{n+1}) \geq \rho(X + X_{n+1}) - \rho(X).$$

Hence if each asset only gets allocated the additional increase in the total portfolio, then the sum of the capital allocated to each asset may not add up to the capital required for the entire portfolio and the sum will never be above. In some cases it may be fair that the last asset, X_{n+1}, gets allocated only the marginal increase and all the other assets get allocated capital assuming that X_{n+1} is not present. But then it is necessary to argue that this particular asset is different from the rest and deserves a special treatment.

In a one-period framework, as pointed out by Fischer [5], where we at time $t = 0$ can evaluate the entire portfolio X and we are not allowed to do any changes before $t = T$ and the variables X and X_{n+1} denote the values at time $t = T$, then it is natural to base the evaluation on (2). Euler capital allocation is only based on the risk as it is evaluated today. The typical situation is often very different. It does not consider that a large part of the portfolio may be invested a long time ago, under different economic conditions and that the corporate may not be in the position to remove part of the portfolio that is not beneficial any more. The present decision is whether to include a new investment in the portfolio. Let X_t and $X_{n+1,t}$ denote the portfolio and the new investment at time t and assume the new investment is a commitment from $t = 0$ to $t = T$. Then it is natural to base the decision on the capital allocation

$$\frac{1}{T}\int_{t=0}^{T} E\{\rho(X_{n+1,t}, X_t + X_{n+1,t})\}dt.$$

In order to evaluate this expression it is necessary to evaluate the future properties of the portfolio and the new investment. This is typically done by simulation. The following schematic example shows how this may be done.

Example 4 Let the portfolio be $X = Z_1 + Z_2$ where one of the Z_j is renewed each year for a period of two years. Each of the variables Z_j may be of the two types X_i, $i = 1, 2$ as defined in Example 3. When we make the investment for a two years period we know the portfolio for the first year, but for the second year the other part of the portfolio is with equal probability equal to X_1 or X_2. Table 1 shows RORAC for the four cases. If the new investment is of type X_2 and the other investment that is lasting for one more year is also of type X_2, then the RORAC

TABLE 2. Average RORAC for a two years period for a new investment depending on the old investment in the portfolio

New investment	Old investment	
	X_1	X_2
X_1	0.031	0.032
X_2	0.064	0.051

for the new investment the first year is 0.044. For the second year will the other investment be of type X_1 or X_2 with equal probability leading to a RORAC equal 0.071 or 0.044 with equal probability. This implies that the expected average annual RORAC for the new investment is equal to (0.044+(0.044+0.071)/2)/2=0.51. By a similar method we may calculate the numbers shown in Table 2. This table may be used in the evaluation of a new investment based on whether the new and the old investment is of type X_1 or X_2.

7. CALCULATION OF CAPITAL ALLOCATION

The easiest method to calculate the Euler capital allocation based on the stochastic properties of the portfolio and a risk measure is to use Monte Carlo simulation. Then it is generated a large number of realizations and the risk measure is calculated by evaluating the realizations empirically. If VaR or ES is used as risk measure with a high value α, we will only use a small part of the simulated values. This makes the simulation inefficient. This may be considerably improved by using importance sampling, see e.g. Kalkbrener et al. [6] or Tasche [13]. If we use Markov chain Monte Carlo (MCMC) it is possible to improve the sampling even more. See Meyn and Tweedie [8] for a thorough introduction to MCMC. In the following example we illustrate how capital allocation may be calculated using Monte Carlo simulation, Importance sampling and MCMC. We give complete algorithms for each type. It is outside the scope of this paper to discuss the different alternatives within each class in detail.

Example 5 Let

$$X^j = \sum_{i=1}^{n} X_i^j$$

denote realization number j of the portfolio. Assume

$$X_i^j = a_i - exp(Y_i^j)$$

where $Y_i^j \sim N(\mu_i, \sigma_i^2)$ and all Y_i^j are uncorrelated in order to simplify the notation. Let $X^{(j)}$ denote the X^j sorted such that

$$X^{(1)} \leq X^{(2)} \leq \cdots \leq X^{(n)}$$

and $X_i^{(j)}$ is the realization for asset i that correspond to $X^{(j)}$. Assume $n_\alpha = (1-\alpha)n$ is an integer such that $-X^{(n_\alpha)}$ is a reasonable estimator for $VaR_\alpha(X)$. Let b be an integer such that $b \leq n_\alpha$.

The Monte Carlo simulation algorithm below gives n realizations of X. From these realizations we find the estimators $VaR_{\alpha,MC}(X)$ for $VaR_\alpha(X)$ and $VaR_{\alpha,MC}(X_i, X)$ for $VaR_\alpha(X_i, X)$.

(1) For $j = 1, 2, \cdots, m$
 (a) Set $X^j = \sum_{i=1}^{n}(a_i - exp(Y_i^j))$
(2) Sort to find $X^{(1)}, \cdots, X^{(m)}$, and the corresponding $X_i^{(j)}$
(3) Set $VaR_{\alpha,MC}(X) = -X^{(n_\alpha)}$
(4) Set $VaR_{\alpha,MC}(X_i, X) = -\frac{1}{2b+1} X^{(n_\alpha)} \sum_{j=-b}^{b} \frac{X_i^{(n_\alpha+j)}}{X^{(n_\alpha+j)}}$

In order to make good estimates for the capital allocation, it is necessary with many realizations estimating this, hence b large. On the other hand it is necessary with n large such that the $2b+1$ realizations $X^{(n_\alpha - b)}, \cdots, X^{(n_\alpha + b)}$ all are sufficiently close to the quantile $VaR_\alpha(X)$. The storage problem may be solved by only storing realizations in an interval $d_1 \leq X^j \leq d_2$ surrounding $VaR_\alpha(X)$.

Only a small portion of the realizations is in the interesting interval. This may be improved by importance sampling. Here is a very simply importance sampling algorithm where $Y^j_{i,IS} \sim N(\mu_{i,IS}, \sigma^2_{i,IS})$. Define $\phi_{\mu,\sigma^2}(y)$ as the density in the $N(\mu, \sigma^2)$ distribution and p^j the importance sampling weight for each realization. Further, define $p^{(j)}$ as the values of p^j sorted according to the size of X^j and scaled such that $0 = p^{(0)} < p^{(1)} < \cdots < p^{(m)} = 1$ and $p^{(j+1)} - p^{(j)} = cp^k$ where p^k is the corresponding value before the sorting. Define $n_{\alpha,IS}$ such that $p^{(n_{\alpha,IS})}$ is closest to $(1-\alpha)n$. We may then define the importance sampler estimators $VaR_{\alpha,IS}(X)$ and $VaR_{\alpha,IS}(X_i, X)$ by the algorithm:

(1) For $j = 1, 2, \cdots, m$
   (a) Set $X^j = \sum_{i=1}^n (a_i - exp(Y^j_i))$
   (b) Set $p^j = \prod_{i=1}^n \frac{\phi_{\mu_i,\sigma_i^2}(Y^i_j)}{\phi_{\mu_{i,IS},\sigma^2_{i,IS}}(Y^i_j)}$
(2) Sort to find $X^{(1)}, \cdots, X^{(m)}$, and the corresponding $X^{(j)}_i$ and $p^{(j)}$.
(3) Set $VaR_{\alpha,MC}(X) = -X^{(n_{\alpha,IS})}$.
(4) Set $VaR_{\alpha,MC}(X_i, X) = -\frac{1}{2b+1} X^{(n_{\alpha,IS})} \sum_{j=-b_{IS}}^{b_{IS}} \frac{X_i^{(n_{\alpha,IS}+j)}}{X^{(n_\alpha+j)}}$.

Still, only a small part of the realizations are used in the estimation and none of these are exactly such that $X = VaR_\alpha(X)$. By using Markov chain Monte Carlo MCMC it is possible to make all realizations equal to $X = VaR_\alpha(X)$ and such that all realizations may be used in the capital allocation. Assume $VaR_\alpha(X)$ is found by another algorithm, for example one of the two algorithms described above. It is necessary with significant larger n in order to find the capital allocation than VaR hence it make sense to use on of the other algorithms for this. Then the MCMC estimator for the capital allocation $VaR_{\alpha,MCMC}(X_i, X)$ may be found by the following algorithm:

(1) Let X^0 be any realization satisfying $X^0 = VaR_\alpha(X)$.
(2) Let $k_1 \leq n$ be any index
(3) For $j = 1, 2, \cdots, m * n$
   (a) Find index $k_2 \leq n$ and $k_1 \neq k_2$ uniformly
   (b) Find $\tilde{Y}^{j+1}_{k_2} \sim N(\mu_{k_2}, \sigma^2_{k_2})$ and $corr(\tilde{Y}^{j+1}_{k_2}, Y^j_{k_2})$ constant
   (c) Set $\tilde{X}^j_{k_2} = (a_{k_2} - exp(Y^j_{k_2}))$
   (d) Set $\tilde{X}^j_{k_1} = X^j_{k_1} + X^j_{k_2} - \tilde{X}^j_{k_2}$
   (e) Set $\tilde{Y}^j_{k_1} = \log(a_{k_1} - \tilde{X}^j_{k_1})$
   (f) Set $p_j = \phi_{\mu_i,\sigma_i^2}(\tilde{Y}^i_j) / \phi_{\mu_i,\sigma_i^2}(Y^i_j)$
   (g) Set $X^{j+1} = \tilde{X}^{j+1}$ with probability $\min\{1, p_j\}$ and else set $X^{j+1} = X^j$.
   (h) Set $k_1 = k_2$
(4) Set $VaR_{\alpha,MCMC}(X_i, X) = -\frac{1}{m} \sum_{j=1}^m X^{jn}_i$.

The MCMC algorithm generates a chain of realizations X^j that are according to the distribution we want to study. But the first realizations in the chain are not from the distribution (a burn-in period) and realizations X^j and X^{j+k} are dependent but the dependence decreases geometrically in k. Table 3 shows the number of numerical operations for the different algorithms. Notice that the number of operations is considerably smaller for MCMC in the capital allocation per realization.

TABLE 3. Number of numerical operations (additions, multiplications, exponential, logarithm) per realization in the three algorithms

	MC	IS	MCMC
$VaR(X)$	3n	6n	-
$VaR(X_i, X)$	$3n/(2b+1)$	$6n/(2b_{IS}+1)$	9

TABLE 4. Estimate for the VaR and the capital allocation using the three algorithms. The standard deviation of the estimates are given en parenthesis.

	MC	IS	MCMC
m	1.000.000	1.000.000	100.000
b	1.600	20.000	-
$VaR_{0.99}(X)$	6.33	6.38	-
$VaR_{0.99}(X_1, X)$	0.038 (0.018)	0.042 (0.016)	0.038 (0.027)
$VaR_{0.99}(X_{31}, X)$	0.064 (0.021)	0.065 (0.016)	0.066 (0.029)
$VaR_{0.99}(X_{61}, X)$	0.109 (0.019)	0.108 (0.021)	0.109 (0.026)

We have tested the three algorithm in an example with $n = 90$ assets with the following parameters: $\sigma_i = 0.5$ and $E\{X_i\} = 0.2$ for all i while the assets are divided into three groups with 30 assets in each group where $\mu_i = 0.44/0.45/0.47$ respectively. The parameters a_i are determined from the other parameters. Table 4 shows the estimate for $VaR_{0.99}(X)$ and the capital allocation $VaR_{0.99}(X_i, X)$ for the portfolio for the three different algorithms. By having 30 equal assets in the portfolio, it is possible to estimate the precision of the different estimates.

In importance sampling we have used $\mu_{i,IS} = \mu_i + 0.2$ resulting in about 10 times as many realizations in an interval close to $VaR_{0.99}(X)$ as in Monte Carlo simulation. This implies that importance sampling may give an improvement with a factor 10 compared to Monte Carlo simulation. In MCMC we have used correlation equal to 0.3 when proposing a new $\tilde{Y}_{k_2}^{j+1}$ value leading to an acceptance ratio at 0.57 and that realizations with X^j and X^{j+5} are almost independent. There is almost no burn-in in this MCMC algorithm. Importance sampling gives 2b=40.000 samples from m= 1 mill. MCMC gives 100.000 samples using about 10 % of the number of operations. This indicates that that MCMC may give an improvement with a factor 100 compared to Monte Carlo simulation. In the test, we have not performed any optimization of the algorithm and parameters. Hence, it is reasonable to assume that it is possible to improve both importance sampling and MCMC even further.

8. Concluding remarks

In this paper we have given an overview over capital allocation using the Euler formula. We have contributed to the research showing that:

- Euler capital allocation with VaR is not monotone even though VaR is monotone. This is shown in both an example with independent discrete variables and with continuous correlated variables.
- The same risk measure should be used for the portfolio and in the capital allocation.
- Simulation may be used in estimating the expected RORAC over the commitment period of an asset.
- Markov chain Monte Carlo may be used in the estimation of the capital allocation.

References

[1] Acerbi, C. and Tasche, D. Expected shortfall: a natural coherent alternative to value at risk, Economic notes, 31(2) 376-388, 2002.

[2] Artzner, P., Delbaen, F., Eber, J.-M. Heath, D., Thinking coherently, Risk 10 (11) 1997.

[3] Artzner, P., Delbaen, F., Eber, J.-M. Heath, D., Coherently measures of risk, Math. Fin. 9 (3) 1999.

[4] Denault, M., Coherent allocation of risk capital, Journal of risk 4(1), 1-34, 2001

[5] Fischer, T., Risk capital allocation by coherent risk measures based on one-side moments, Insurance: Mathematics and Economics, 32(1), 135-146, 2003

[6] Kalkbrener, M., Lotter, H., Overbeck, L. Sensible and efficient capital allocation for credit portfolios, RISK 17(1):S19:24, 2004.

[7] Litterman, R., Hot spots and hedges, The Journal of portfolio management, 22, 52-75, 1996.

[8] Meyn, S. O and Tweedie, R. L., Markov chains and Stochastic stability, Springer-Verlag, 1993.

[9] Tasche, D., Risk contribution and performance measurements, working paper, Technische Universitet Munchen, 1999.

[10] Tasche, D., Expected shortfall and beyond, Journal of banking and finance 26(7) 1519-1533, 2002.

[11] Tasche, D., Capital Allocation with CreditRisk+, In: CreditRisk+ in the Banking Industry, V.M. Gundlach and F.B. Lehrbass (eds.), pp 25-44. Springer, 2004,

[12] Tasche, D., Allocating portfolio economic capital to sub-portfolio, In: Economical capital: a practitioners guide A. Dev (ed.), Risk books, pp. 273-302, 2004.

[13] Tasche, D., Measuring sectoral diversification in an asymptotic multi-factor framework, Journal of Credit risk 2(3) 33-55 (2006)

[14] Tasche, D. Euler allocation: Theory and practice, Working paper, 2007.

[15] Urban, M, Dittrich, J., Kluppelberg, C., and Stolting, R., Allocation of risk capital to insurance portfolios, Blatter der DGVFM, 26, 389-406, 2004

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